# The Lagrange Points

There are five equilibrium points to be found in the vicinity of two orbiting masses. They are called *Lagrange Points* in honour of the French-Italian mathematician Joseph Lagrange, who discovered them while studing the restricted three-body problem. The term "restricted" refers to the condition that two of the masses are very much heavier than the third. Today we know that the full three-body problem is chaotic, and so cannot be solved in closed form. Therefore, Lagrange had good reason to make some approximations. Moreover, there are many examples in our solar system that can be accurately described by the restricted three-body problem.

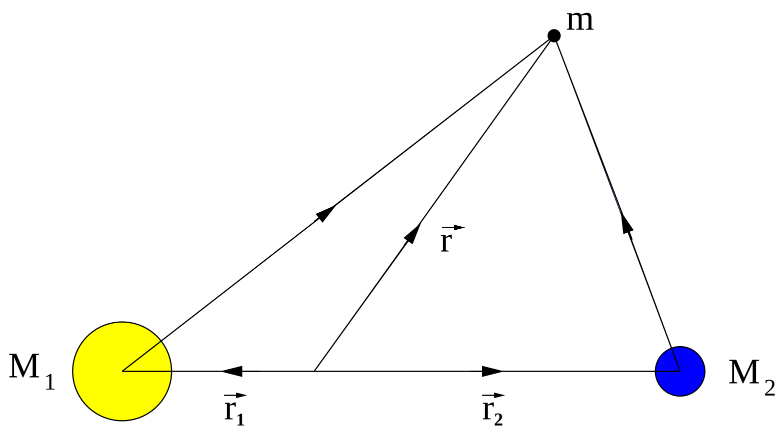


Figure 1: The restricted three-body problem

The procedure for finding the Lagrange points is fairly straightforward: We seek solutions to the equations of motion which maintain a constant separation between the three bodies. If $M_1$ and $M_2$ are the two masses, and $\vec{r}_1$ and $\vec{r}_2$ are their respective positions, then the total force exerted on a third mass $m$, at a position $\vec{r}$, will be

$$\vec{F} = -\frac{GM_1 m}{|\vec{r} - \vec{r}_1|^3}(\vec{r} - \vec{r}_1) - \frac{GM_2 m}{|\vec{r} - \vec{r}_2|^3}(\vec{r} - \vec{r}_2)\,. \tag{1}$$

The catch is that both $\vec{r}_1$ and $\vec{r}_2$ are functions of time since $M_1$ and $M_2$ are orbiting each other. Undaunted, one may proceed and insert the orbital solution for $\vec{r}_1(t)$ and $\vec{r}_2(t)$ (obtained by solving the two-body problem for $M_1$ and $M_2$) and look solutions to the equation of motion

$$\vec{F}(t) = m\frac{d^2\vec{r}(t)}{dt^2}\,, \tag{2}$$

that keep the relative positions of the three bodies fixed. It is these *stationary* solutions that are know as Lagrange points.

The easiest way to find the stationary solutions is to adopt a co-rotating frame of reference in which the two large masses hold fixed positions. The new frame of reference has its origin at the center of mass, and an angular frequency $\Omega$ given by Kepler's law:

$$\Omega^2 R^3 = G(M_1 + M_2)\,. \tag{3}$$

Here $R$ is the distance between the two masses. The only drawback of using a non-inertial frame of reference is that we have to append various pseudo-forces to the equation of motion. The effective force in a frame rotating with angular velocity $\vec{\Omega}$ is related to the inertial force $\vec{F}$ according to the transformation

$$\vec{F}_\Omega = \vec{F} - 2m\left(\vec{\Omega} \times \frac{d\vec{r}}{dt}\right) - m\,\vec{\Omega} \times (\vec{\Omega} \times \vec{r})\,. \tag{4}$$

The first correction is the coriolis force and the second is the centrifugal force. The effective force can be derived from the generalised potential

$$U_\Omega = U - \vec{v} \cdot (\vec{\Omega} \times \vec{r}) + \frac{1}{2}(\vec{\Omega} \times \vec{r}) \cdot (\vec{\Omega} \times \vec{r})\,, \tag{5}$$

as the generalised gradient

$$\vec{F}_\Omega = -\nabla_{\vec{r}} U_\Omega + \frac{d}{dt}\left(\nabla_{\vec{v}} U_\Omega\right)\,. \tag{6}$$

The velocity dependent terms in the effective potential do not influence the positions of the equilibrium points, but the are crucial in determining the dynamical stability of motion about the equilibrium points. A plot of $U_\Omega$ with $\vec{v} = 0$, $M_1 = 10M_2 = 1$ and $R = 10$ is shown in Figure 2. The extrema of the generalised potential are labeled L1 through L5.

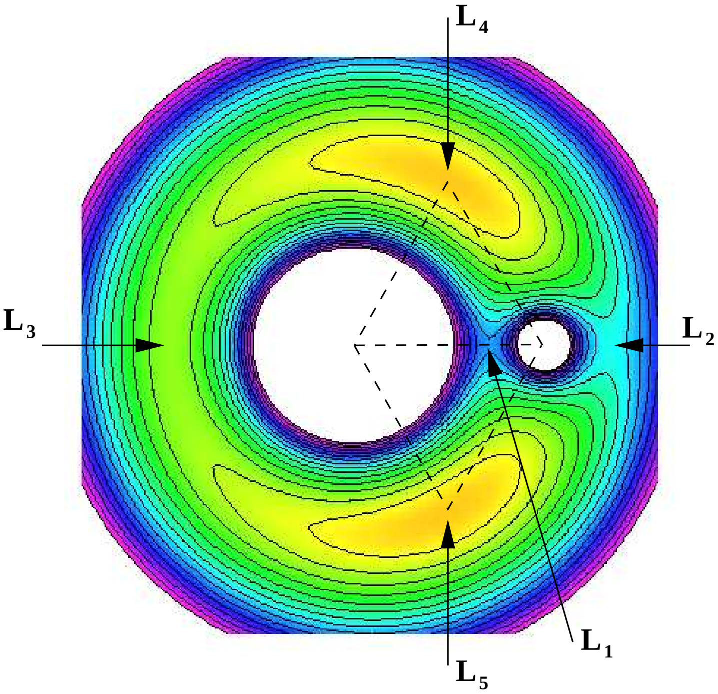


Figure 2: A countour plot of the generalised potential.

Choosing a set of cartesian coordinates originating from the center of mass with the $z$ axis alined with the angular velocity, we have

$$
\begin{aligned}
\vec{\Omega} &= \Omega\,\hat{k} \\
\vec{r} &= x(t)\,\hat{\imath} + y(t)\,\hat{\jmath} \\
\vec{r}_1 &= -\alpha R\,\hat{\imath} \\
\vec{r}_2 &= \beta R\,\hat{\imath}
\end{aligned} \tag{7}
$$

where

$$
\alpha = \frac{M_2}{M_1 + M_2}\,, \quad \beta = \frac{M_1}{M_1 + M_2}\,. \tag{8}
$$

To find the static equilibrium points we set the velocity $\vec{v} = d\vec{r}/dt$ to zero and seek solutions to the equation $\vec{F}_\Omega = \vec{0}$, where

$$
\begin{aligned}
\vec{F}_\Omega \;=\; & \Omega^2 \left( x - \frac{\beta(x+\alpha R)R^3}{((x+\alpha R)^2 + y^2)^{3/2}} - \frac{\alpha(x-\beta R)R^3}{((x-\beta R)^2 + y^2)^{3/2}} \right) \hat{\imath} \\
& \Omega^2 \left( y - \frac{\beta y R^3}{((x+\alpha R)^2 + y^2)^{3/2}} - \frac{\alpha y R^3}{((x-\beta R)^2 + y^2)^{3/2}} \right) \hat{\jmath}\,.
\end{aligned} \tag{9}
$$

Here the mass $m$ has been set equal to unity without loss of generality. The brute-force approach for finding the equilibrium points would be to set the magnitude of each force component to zero, and solve the resulting set of coupled, fourteenth order equations for $x$ and $y$. A more promising approach is to think about the problem physically, and use the symmetries of the system to guide us to the answer.

Since the system is reflection-symmetric about the $x$-axis, the $y$ component of the force must vanish along this line. Setting $y = 0$ and writing $x = R(u + \beta)$ (so that $u$ measures the distance from $M_2$ in units of $R$), the condition for the force to vanish along the x-axis reduces to finding solutions to the three fifth-order equations

$$u^2((1-s_1)+3u+3u^2+u^3) = \alpha(s_0+2s_0u+(1+s_0-s_1)u^2+2u^3+u^4)\,, \tag{10}$$

where $s_0 = \mathrm{sign}(u)$ and $s_1 = \mathrm{sign}(u+1)$. The three cases we need to solve have $(s_0, s_1)$ equal to $(-1, 1)$, $(1, 1)$ and $(-1, -1)$. The case $(1, -1)$ cannot occur. In each case there is one real root to the quintic equation, giving us the positions of the first three Lagrange points. We are unable to find closed-form solutions to equation (10) for general values of $\alpha$, so instead we seek approximate solutions valid in the limit $\alpha \ll 1$. To lowest order in $\alpha$, we find the first three Lagrange points to be positioned at

$$\begin{aligned}
L1: &\quad \left(R\left[1-\left(\frac{\alpha}{3}\right)^{1/3}\right], 0\right)\,, \\
L2: &\quad \left(R\left[1+\left(\frac{\alpha}{3}\right)^{1/3}\right], 0\right)\,, \\
L3: &\quad \left(-R\left[1+\frac{5}{12}\alpha\right], 0\right)\,.
\end{aligned} \tag{11}$$

For the earth-sun system $\alpha \approx 3 \times 10^{-6}$, $R = 1\,\mathrm{AU} \approx 1.5 \times 10^8\,\mathrm{km}$, and the first and second Lagrange points are located approximately 1.5 million kilometers from the earth. The third Lagrange point - home of the mythical planet X - orbits the sun just a fraction further out than the earth.

Identifying the remaining two Lagrange points requires a little more thought. We need to balance the centrifugal force, which acts in a direction radially outward from the center of mass, with the gravitational force exerted by the two masses. Clearly, force balance in the direction perpendicular to centrifugal force will only involve gravitational forces. This suggests that we should resolve the force into directions parallel and perpendicular to $\vec{r}$. The appropriate projection vectors are $x\hat{\imath} + y\hat{\jmath}$ and $y\hat{\imath} - x\hat{\jmath}$. The perpendicular projection yields

$$F_\Omega^\perp = \alpha\beta y \Omega^2 R^3 \left( \frac{1}{((x - R\beta)^2 + y^2)^{3/2}} - \frac{1}{((x + R\alpha)^2 + y^2)^{3/2}} \right) . \tag{12}$$

Setting $F_\Omega^\perp = 0$ and $y \neq 0$ tells us that the equilibrium points must be equidistant from the two masses. Using this fact, the parallel projection simplifies to read

$$F_\Omega^{||} = \Omega^2 \frac{x^2 + y^2}{R} \left( \frac{1}{R^3} - \frac{1}{((x - R\beta)^2 + y^2)^{3/2}} \right) . \tag{13}$$

Demanding that the parallel component of the force vanish leads to the condition that the equilibrium points are at a distance $R$ from each mass. In other words, L4 is situated at the vertex of an equilateral triangle, with the two masses forming the other vertices. L5 is obtained by a mirror reflection of L4 about the $x$-axis. Explicitly, the fourth and fifth Lagrange points have coordinates

$$\begin{aligned} L4: &\quad \left( \frac{R}{2} \left( \frac{M_1 - M_2}{M_1 + M_2} \right), \frac{\sqrt{3}}{2} R \right) , \\ L5: &\quad \left( \frac{R}{2} \left( \frac{M_1 - M_2}{M_1 + M_2} \right), -\frac{\sqrt{3}}{2} R \right) . \end{aligned} \tag{14}$$

# Stability Analysis

Having established that the restricted three-body problem admits equilibrium points, our next task is to determine if they are stable. Usually it is enough to look at the shape of the effective potential and see if the equilibrium points occur at hills, valleys or saddles. However, this simple criterion fails when we have a velocity dependent potential. Instead, we must perform a linear stability analysis about each Lagrange point. This entails linearising the equation of motion about each equilibrium solution and solving for small departures from equilibrium. Writing

$$\begin{aligned} x &= x_i + \delta x\,, \quad v_x = \delta v_x\,, \\ x &= y_i + \delta y\,, \quad v_y = \delta v_y\,, \end{aligned} \tag{15}$$

where $(x_i, y_i)$ is the position of the $i$-th Lagrange point, the linearised equations of motion become

$$\frac{d}{dt}\begin{pmatrix} \delta x \\ \delta y \\ \delta v_x \\ \delta v_y \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \dfrac{d^2\,U_\Omega}{dx^2} & \dfrac{d^2\,U_\Omega}{dxdy} & 0 & 2\Omega \\ \dfrac{d^2\,U_\Omega}{dydx} & \dfrac{d^2\,U_\Omega}{dy^2} & -2\Omega & 0 \end{pmatrix} \begin{pmatrix} \delta x \\ \delta y \\ \delta v_x \\ \delta v_y \end{pmatrix}. \tag{16}$$

Here the second derivatives of $U_\Omega$ are evaluated at $\vec{r} = (x_i, y_i)$.

## Stability of L1 and L2

The stability of the first and second Lagrange points of the earth-sun system is an important consideration for some NASA missions. Currently the solar observatory SOHO is parked at L1, and NASA plans to send the Microwave Anisotropy Probe (MAP) out to L2. It has also be suggested that the Next Generation Space Telescope (NGST) should be positioned at L2.

The curvature of the effective potential near L1 and L2 reveals them to be saddle points:

$$\frac{d^2\,U_\Omega}{dx^2} = \mp 9\Omega^2\,, \quad \frac{d^2\,U_\Omega}{dy^2} = \pm 3\Omega^2\,, \quad \frac{d^2\,U_\Omega}{dxdy} = \frac{d^2\,U_\Omega}{dydx} = 0\,. \tag{17}$$

Solving for the eigenvalues of the linearised evolution matrix we find

$$\lambda_\pm = \pm\Omega\sqrt{1+2\sqrt{7}} \quad \text{and} \quad \sigma_\pm = \pm i\Omega\sqrt{2\sqrt{7}-1}\,. \tag{18}$$

The presence of a positive, real root tells us that L1 and L2 are dynamically unstable. Small departures from equilibrium will grow exponentially with a e-folding time of

$$\tau = \frac{1}{\lambda_+} \approx \frac{2}{5\Omega}\,. \tag{19}$$

For the earth-sun system $\Omega = 2\pi\,\mathrm{year}^{-1}$ and $\tau \approx 23$ days. In other words, a satellite parked at L1 or L2 will wander off after a few months unless course corrections are made.

## Stability of L3

A popular theme in early science fiction stories was invasion by creatures from Planet X. The requirement that Planet X remain hidden behind the sun places it at the L3 point of the earth-sun system. Unfortunately for our would-be invaders, the L3 point is a weak saddle point of the effective potential with curvature

$$\frac{d^2\,U_\Omega}{dx^2} = -3\Omega^2\,, \quad \frac{d^2\,U_\Omega}{dy^2} = \frac{7M_2}{8M_1}\,\Omega^2\,, \quad \frac{d^2\,U_\Omega}{dxdy} = \frac{d^2\,U_\Omega}{dydx} = 0\,. \tag{20}$$

To leading order in $M_2/M_1$, the eigenvalues of the linearised evolution matrix are

$$\lambda_\pm = \pm\Omega\sqrt{\frac{3M_1}{8M_2}} \quad \text{and} \quad \sigma_\pm = \pm i\Omega\,\sqrt{7}\,. \tag{21}$$

The real, positive eigenvalue spells disaster for Planet X. Its orbit is exponentially unstable, with an e-folding time of roughly $\tau = 150$ years. While there can be no Planet X, the long e-folding time makes L3 a good place to park your invasion force while final preparations are made...

## Stability of L4 and L5

The stability analysis around L4 and L5 yields something of a suprise. While these points correspond to local maxima of the generalised potential - which usually implies a state of unstable equilibrium - they are in fact stable. Their stability is due to the coriolis force. Initially a mass situated near L4 or L5 will tend to slide down the potential, but as it does so it picks up speed and the coriolis force kicks in, sending it into an orbit around the Lagrange point. The effect is analogous to how a hurricane forms on the surface of the earth: as air rushes into a low pressure system it begins to rotate because of the coriolis force and a stable vortex is formed. Explicitly, the curvature of the potential near L4 is given by

$$\frac{d^2\,U_\Omega}{dx^2} = \frac{3}{4}\Omega^2\,, \quad \frac{d^2\,U_\Omega}{dy^2} = \frac{9}{4}\Omega^2\,, \quad \frac{d^2\,U_\Omega}{dxdy} = \frac{d^2\,U_\Omega}{dydx} = \frac{3\sqrt{3}}{4}\kappa\Omega^2\,, \tag{22}$$

where $\kappa = (M_1 - M_2)/(M_1 + M_2)$. The eigenvalues of the linearised evolution matrix are found to equal

$$\begin{aligned} \lambda_\pm &= \pm i\,\frac{\Omega}{2}\sqrt{2 - \sqrt{27\kappa^2 - 23}} \\ \sigma_\pm &= \pm i\,\frac{\Omega}{2}\sqrt{2 + \sqrt{27\kappa^2 - 23}}\,. \end{aligned} \tag{23}$$

The L4 point will be stable if the eigenvalues are pure imaginary. This will be true if

$$\kappa^2 \geq \frac{23}{27} \quad \text{and} \quad \sqrt{27\kappa^2 - 23} \leq 2\,. \tag{24}$$

The second condition is always satisfied, while the first requires

$$M_1 \geq 25M_2\left(\frac{1+\sqrt{1-4/625}}{2}\right)\,. \tag{25}$$

When the L4 and L5 points yield stable orbits they are refered to as Trojan points after the three Trojan asteroids, Agamemnon, Achilles and Hector, found at the L4 and L5 points of Jupiter's orbit. The mass ratios in the earth-sun and earth-moon system are easily large enough for their L4 and L5 points to be home to Trojan satellites, though none have been found.

## Author

These notes were written by Neil J. Cornish with input from Jeremy Goodman.